Analysis I Lecture 7

Definition Argument of z is $\varphi \in \mathbb{R}$

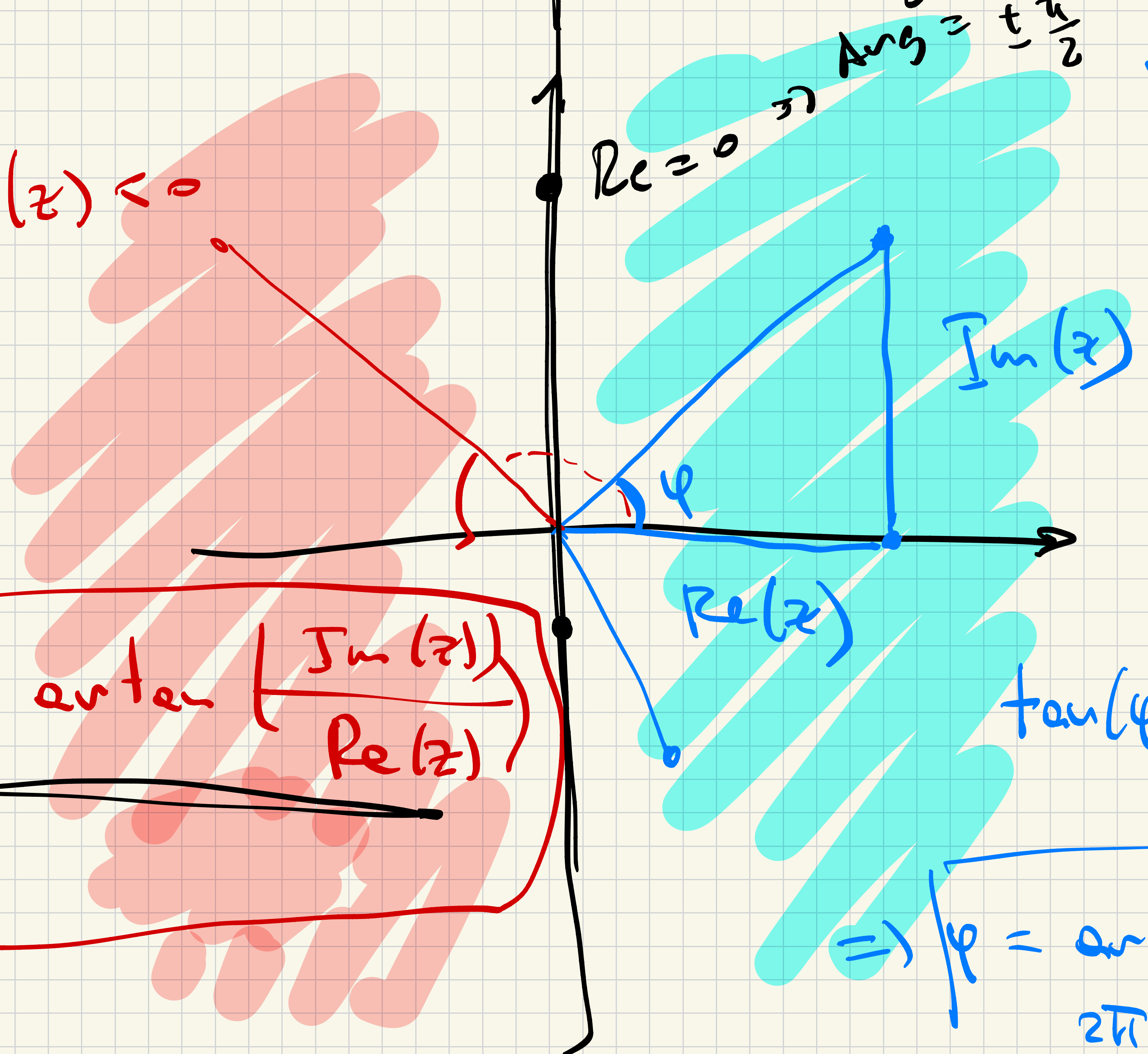
s.t. $z = |z| \cdot (\cos\varphi + i \sin\varphi)$.

We denote by $\mathrm{Arg}(z)$.

sin, cos are 2π-periodic

so there are different choices for such φ.

How to compute argument

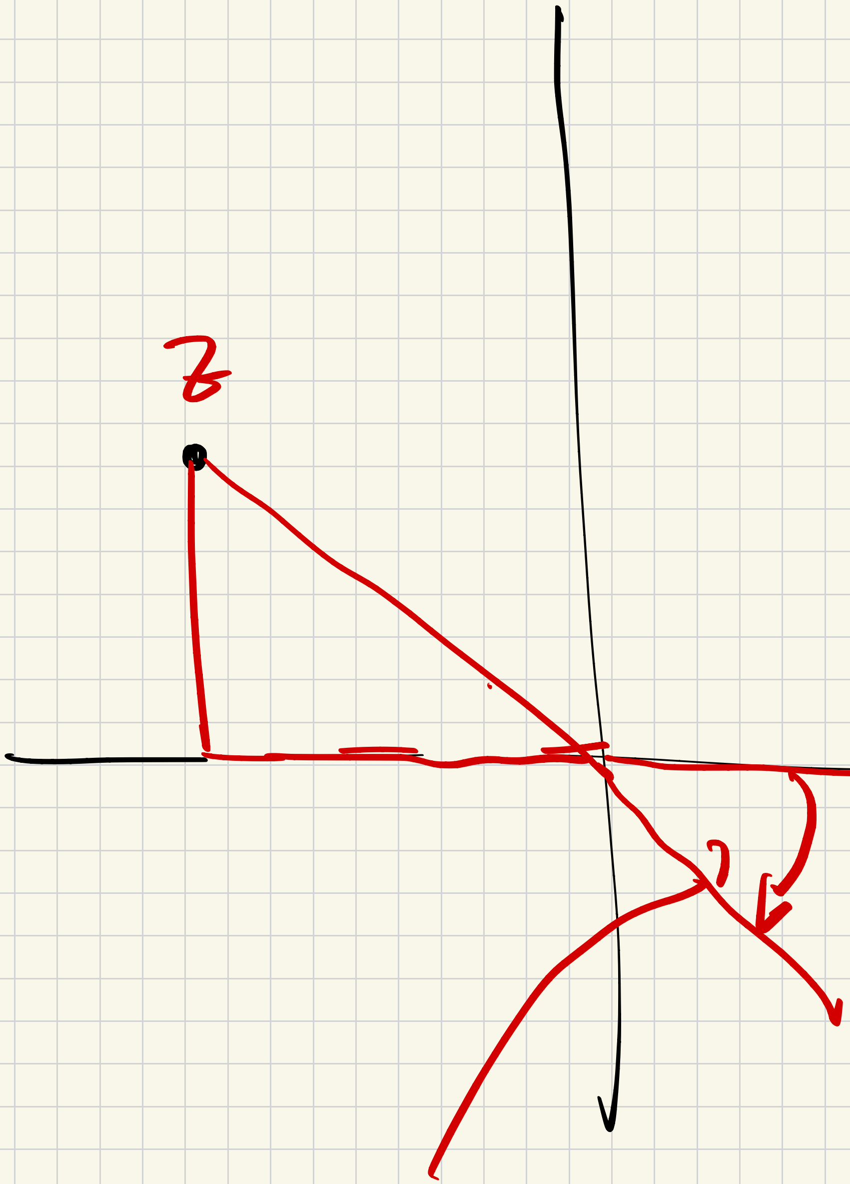


$$\tan(\varphi) = \frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}$$

$$\Rightarrow \boxed{\varphi = \arctan\left(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right)}$$

$2\pi \cdot k$

$$\boxed{\varphi = \pi + \arctan\left(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right)}$$



$\mathrm{Im}(z) > 0$

$\mathrm{Re}(z) < 0$

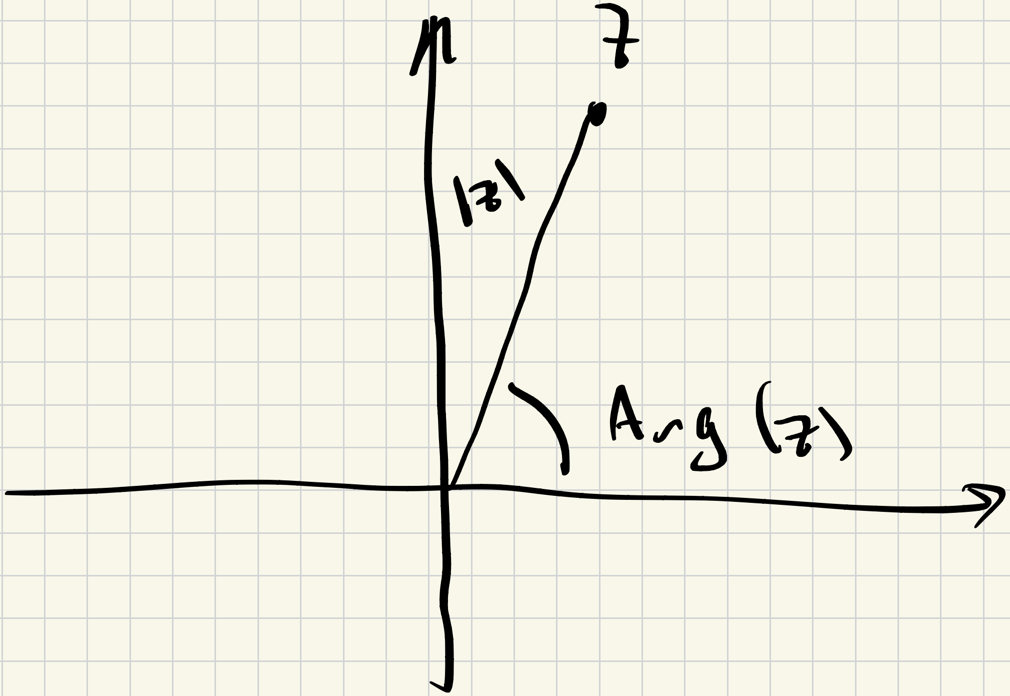
$$\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)} < 0$$

$$-\frac{\pi}{2} < \arctan\left(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right) < 0$$

Polar presentation!

For any $z \in \mathbb{C}$ we have;

$$z = |z| \cdot \left(\cos(\operatorname{Arg} z) + i \sin(\operatorname{Arg} z) \right)$$



Product in the polar form

$$z_1 \cdot z_2 = \boxed{|z_1| \cdot |z_2|} \cdot \left(\cos\left(\underline{\underline{\operatorname{Arg} z_1 + \operatorname{Arg} z_2}}\right) + i \sin\left(\underline{\underline{\operatorname{Arg} z_1 + \operatorname{Arg} z_2}}\right)\right)$$

Euler formula

Definition Complex exponential function

$$e^{z} = e^{x+iy} := \boxed{e^{x}} \cdot \left(\cos \underline{\underline{y}} + i \sin \underline{\underline{y}}\right)$$

In other words,

$$e^{z} = e^{\mathrm{Re}(z)} \cdot \left(\cos(\mathrm{Im}\, z) + i \sin(\mathrm{Im}\, z)\right)$$

Example Euler's identity

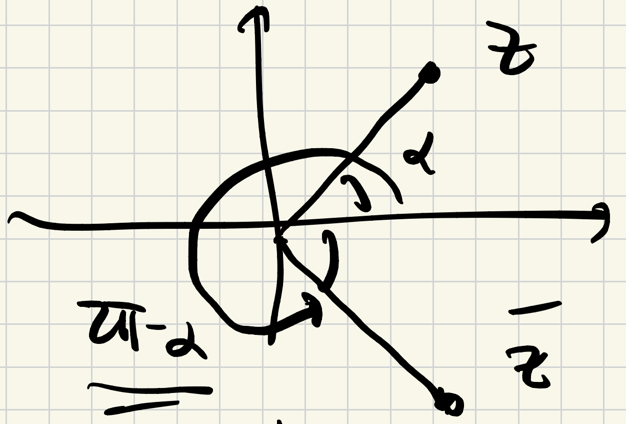
$$\boxed{e^{\pi i} = -1}$$

Proposition

$$e^z = \underbrace{e^{\mathrm{Re}(z)}}_{|e^z|} \cdot \left(\cos(\underbrace{\mathrm{Im}\, z}_{\mathrm{Arg}(e^z)}) + i\sin(\underbrace{\mathrm{Im}\, z}_{\mathrm{Arg}(e^z)})\right)$$

- $|e^z| = e^{\mathrm{Re}(z)}$

- $\mathrm{Arg}(e^z) = \mathrm{Im}(z)$



- $\overline{e^z} = e^{\boxed{\bar z}}$: taking complex conjugate is equivalent to changing $\mathrm{Arg}(z)$ to $-\mathrm{Arg}(z)$

Proposition (cont.)

$$e^{z} = \underbrace{e^{\operatorname{Re}(z)}}\cdot\left(\cos(\operatorname{Im} z) + i\sin(\operatorname{Im}(z))\right)$$

- $e^{(z+2\pi i)} = e^{z}$
- $e^{z_1} = e^{z_2} \iff z_1 = z_2 + 2\pi i \cdot k$ $\quad k \in \mathbb{Z}$

sin, cos are 2π-periodic

$$\operatorname{Im}(z + 2\pi i) = \operatorname{Im}(z) + 2\pi$$

$$\operatorname{Re}(z + 2\pi i) = \operatorname{Re}(z)$$

to see when $e^{z_1} = e^{z_2}$

Notice 1) $|e^{z_1}| = |e^{z_2}| \Rightarrow$

$$e^{\mathrm{Re}(z_1)} = e^{\mathrm{Re}(z_2)} \Rightarrow \mathrm{Re}(z_1) = \mathrm{Re}(z_2$$

2) $\cos(\mathrm{Im}(z_1)) + i\sin(\mathrm{Im}(z_1)) = \cos(\mathrm{Im}(z_2)) + i\sin(\mathrm{Im}\, z_2$

$$\Rightarrow \mathrm{Im}(z_1) = \mathrm{Im}(z_2) + 2\pi k \quad k \in \mathbb{Z}$$

$\blacksquare$

Proposition (cont.)

- $e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$

$n \in \mathbb{Z}$

- $(e^{z})^{n} = e^{n \cdot z} = \underbrace{e^{z} \cdot e^{z} \cdots e^{z}}_{n \text{ times}} = (e^{z})^{n}$

To show this use polar form and product formula in the polar form.

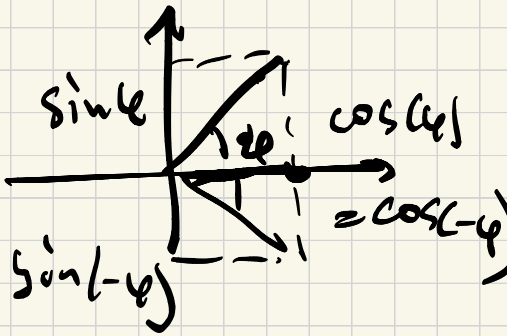
Theorem (Euler's formula)

$\forall \varphi \in \mathbb{R}$ we have:

$$\cos(\varphi) = \frac{e^{i\varphi} + e^{-i\varphi}}{2}$$

$$\sin(\varphi) = \frac{e^{i\varphi} - e^{-i\varphi}}{2i}$$

Proof for cos:



$$\theta = i\varphi$$

$$\frac{e^{i\varphi} + e^{-i\varphi}}{2} = \frac{\overset{1}{\cancel{e^0}}\cdot\left(\cos(\varphi) + i\sin\varphi\right) + \underset{1}{\cancel{e^0}}\left(\cos(-\varphi) + i\sin(-\varphi)\right)}{2}$$

$$\sin(-\varphi) = -\sin\varphi$$

$$\cos(-\varphi) = \cos(\varphi)$$

$$= \frac{2\cos(\varphi) + i\,\overset{0}{\cancel{(\sin\varphi - \sin\varphi)}}}{2} = \cos(\varphi)$$

■

Theorem de Moivre's formula

For any $\varphi \in \mathbb{R}$ $n \in \mathbb{N}$ we have

$$\underbrace{(\cos\varphi + i\sin\varphi)}_{z}{}^{n} = \underbrace{\cos(n\varphi) + i\sin(n\varphi)}_{z^n}$$

Proof

$$|z^n| = |z|^n = \left(\sqrt{\cos^2\varphi + \sin^2\varphi}\right)^n = 1^n = 1$$

$$\arg(z^n) = n \cdot \arg(z) = n \cdot \varphi \Rightarrow z^n = \cos(n\varphi) + i\sin(n\varphi)$$

Different proof:

$$\cos(\varphi) + i\sin(\varphi) = e^{i\varphi}$$

$$\Vert$$

$$\left(\quad \underline{\qquad} + \underline{\qquad} \right)^n = \left(e^{i\varphi}\right)^n = e^{in\varphi}$$

$$= \cos(n \cdot \varphi) + i\sin(n \cdot \varphi)$$

$$\blacksquare$$

Example Application of de Moivre's formule

$$\cos(3\cdot\varphi) = 4\cos^3\varphi - 3\cos(\varphi)$$

We can get this formula by using de Moivres:

(

$$\left(\cos(\varphi)+ i\sin(\varphi)\right)^3 = \cos(3\varphi)+ i\sin(3\varphi)$$

$$+$$

$$\left(\cos(\underset{\varphi}{\cancel{-}\varphi}) \overset{-}{+} i\sin(\cancel{+}\varphi)\right)^3 = \cos(\underset{3\varphi}{-\cancel{3}\varphi}) \overset{-}{+} i\sin(\cancel{+}3\varphi)$$

$$\left(2\cos^3\varphi - 6\cos\varphi\,\underset{1-\cos^2\varphi}{\sin^2\varphi}\right) \underset{\parallel}{=} 2\cos(3\varphi)$$

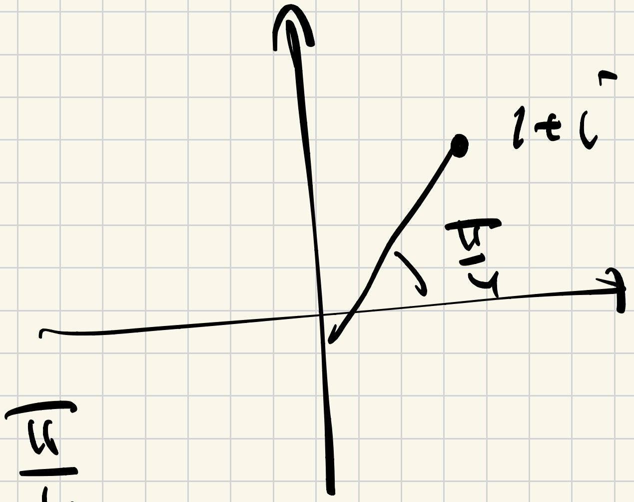
$$\boxed{4\cos^3\varphi - 3\cos(\varphi) = \cos(3\varphi)}$$

[redacted]

Example

Compute $(1+i)^8$.

$$(1+i)^8 = 16$$



$$|1+i| = \sqrt{2}, \quad \mathrm{Arg}(1+i) = \frac{\pi}{4}$$

$$(1+i) = \sqrt{2} \cdot \left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4} \right) \Rightarrow$$

$$\Rightarrow (1+i)^8 = (\sqrt{2})^8 \left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4} \right)^8 =$$

$$= 16 \cdot \left(\underbrace{\cos \overset{2\pi}{\cancel{8 \cdot \frac{\pi}{4}}}}_{1} + i \underbrace{\sin \overset{2\pi}{\cancel{8 \cdot \frac{\pi}{4}}}}_{0} \right) = 16 \cdot (1 + i \cdot 0) = 16$$

Solving polynomial equations in $\mathbb{C}$.

Quadratic formula

$$ax^2 + bx + c = 0$$

$$\text{discriminant} \longrightarrow \Delta = b^2 - 4ac$$

$$x_1 = \frac{-b + \sqrt{\Delta}}{2a}$$

$$x_2 = \frac{-b - \sqrt{\Delta}}{2a}$$

$\Rightarrow$) If $\Delta > 0$ the there 2 distinct real roots

---

If $\Delta = 0$ the $ax^2 + bx + c = a \cdot (x-\lambda)^2$ for some $\lambda \in \mathbb{R}$

$\Rightarrow$ 1 root of multiplicity 2.

If $\Delta < 0$ then no real roots

But if we treat $\sqrt{\Delta}$ as complex number we have root even in case when $\Delta < 0$.

So we have 2 complex solutions

Roots of complex numbers

Proposition For every $z \in \mathbb{C} \setminus \{0\}$ there exist distinct $w_1 \dots w_n \in \mathbb{C} \setminus \{0\}$ such that $w_i^n = z$.

In other words roots always exist.
and there is n different choices for $\sqrt[n]{z}$ if $z \neq 0$.

How to find roots?

Example Let $\sqrt{i} = x + iy$ then

$$\left.\begin{matrix}(x+iy)^2 = i \\ \shortparallel \\ (x^2-y^2) + 2xy\cdot i\end{matrix}\right\} \Rightarrow \begin{cases} x^2-y^2 = 0 \\ 2x\cdot y = 1 \end{cases}$$

$$\begin{cases} x^2 - y^2 = 0 \\ 2x \cdot y = 1 \end{cases}$$

$$\xrightarrow{\quad} (x-y)(x+y) = 0$$

$$\Rightarrow \left[\begin{array}{l} x = -y \\ x = y \end{array}\right.$$

$x = y$

$x = -y$

$$2x^2 = 1$$

$$\Uparrow$$

$$y = x = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$$

$$\left|\begin{array}{l} 2x \cdot (-x) = 1 \\ -x^2 = \frac{1}{2} \\ x^2 = -\frac{1}{2} \\ \text{No real solutions} \end{array}\right.$$

So we get

$$\sqrt{i} = x + iy = \pm\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot i\right)$$

∎

A better way using polar form

If $w^n = z$ then

$$\begin{cases} |w|^n = |z| \\ n \cdot \operatorname{Arg}(w) = \operatorname{Arg}(z) + 2\pi \cdot k \end{cases}$$

$k \in \mathbb{Z}$

this means that:

$$W = \sqrt[n]{|z|}\left(\cos\left(\frac{\arg(z)}{n}\right) + i\sin\left(\frac{\arg(z)}{n}\right)\right)$$

$|w|$

There a lot of choices for $\frac{\arg(z)}{n}$

And this choices give different values for w

Example $w^2 = i$

$$\Rightarrow \quad |w| = \sqrt{|i|} = \sqrt{1} = 1$$

$$\operatorname{Arg} w = \frac{\operatorname{Arg}(i)}{2} = \begin{cases} \frac{\pi}{2}/2 = \frac{\pi}{4} \\ \left(\frac{\pi}{2} + 2\pi\right)/2 = \frac{5\pi}{4} \end{cases}$$

Notice

$$\frac{\frac{\pi}{2} + 2(2\pi)}{2} = \frac{\pi}{4} + 2\pi$$

In general the roots $w_1 \dots w_n$ of order n of complex number $z \neq 0$ are given by:

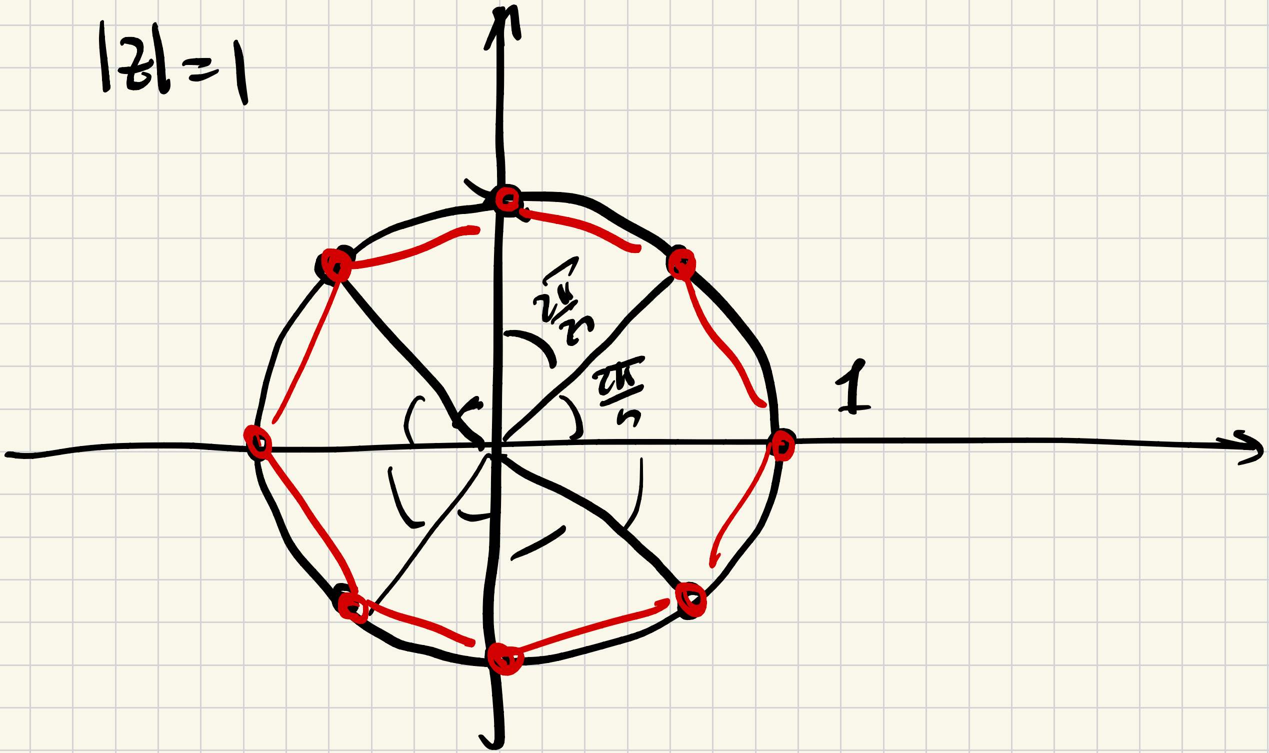
$$W_k = \sqrt[n]{|Z|}\left(\cos\left(\frac{\operatorname{Arg} z}{n} + k\cdot\frac{2\pi}{n}\right) + i\sin\left(\frac{\operatorname{Arg} z}{n} + k\cdot\frac{2\pi}{n}\right)\right),$$

$k = 1 \dots n$

$$n \cdot \left(\frac{\text{Arg } z}{n} + k \cdot \frac{2\pi}{n} \right) = \text{Arg}(z) + k \cdot 2\pi$$

$|z| = 1$

Pictorially



$\sqrt[n]{1}$

$w_1^5 = w_2^5 = \dots = w_5^5$